

# Models for Unsaturated Hydraulic Conductivity Based on Truncated Lognormal Pore-size Distributions

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## Abstract

We develop a closed-form three-parameter model for unsaturated hydraulic conductivity associated with a three-parameter lognormal model of moisture retention, which is based on lognormal grainsize distribution. The derivation of the model is made possible by a slight modification to the theory of Mualem. We extend the three-parameter lognormal distribution to a four-parameter model that also truncates the pore size distribution at a minimum pore radius. We then develop the corresponding four-parameter model for moisture retention and the associated closed-form expression for unsaturated hydraulic conductivity. The four-parameter model is fitted to experimental data, similar to the models of Kosugi and van Genuchten. The proposed four-parameter model retains the physical basis of Kosugi's model, while improving fit to observed data especially when simultaneously fitting pressure-saturation and pressure-conductivity data.

*Keywords:* lognormal distribution, moisture retention model, unsaturated hydraulic conductivity

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## 1. Introduction

Kosugi (1994) assumed pore size is a lognormal random variable and derived a three-parameter model for moisture retention, the three parameters being the mean and variance of the pore-size distribution and the maximum pore radius. In the limiting case where the maximum pore radius approaches infinity the three-parameter model simplifies to a two-parameter model for which Kosugi (1996) developed the closed-form expression for unsaturated hydraulic conductivity using the theory of Mualem (1976). Kosugi (1996) did not develop a three-parameter closed-form equation for hydraulic conductivity, but reverted to the two-parameter form, owing to difficulty in analytically integrating the expression of Mualem (1976). We extend the work of Kosugi (1994, 1996) and develop closed-form expressions for unsaturated hydraulic conductivity associated with the three-parameter lognormal moisture retention model. The derivation of the closed-form equation for unsaturated hydraulic conductivity is made possible by a slight modification to the theory of Mualem (1976). Further, we modify the pore-size probability density function (PDF) of Kosugi (1994) by incorporating a nonzero minimum pore radius, as suggested by Brutsaert (1966). This modification results

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in a four-parameter moisture retention model and a corresponding four-parameter closed-form equation for unsaturated hydraulic conductivity, again obtained using the modified theory of Mualem (1976). The four parameters in the proposed model are all based on physical properties of the porous medium (similar to the model of Kosugi (1996)), not fitting parameters without physical significance.

## 2. Theory

The lognormal distribution is commonly used to statistically characterize pore size in granular porous media. Brutsaert (1966) and Kosugi (1994, 1996) considered lower- and upper-tail truncated lognormal PDFs for pore-size distributions. Brutsaert (1966) considered the log-transformed random pore radius  $R - r_0$ , where  $r_0$  is the radius at which the effective moisture content vanishes (associated with residual saturation).

### 2.1. The three-parameter lognormal model

The classical (non-truncated) lognormal distribution for pore size (Brutsaert, 1966), measured here by the random pore radius  $R \in [0, \infty]$ . For a real porous medium  $R \in [0, r_{\max}]$ , where  $r_{\max}$  is some finite maximum pore radius. To correct for the finite interval, Kosugi (1994) used the random variable  $R_e = (1/R - 1/r_{\max})^{-1}$  to rescale the classical PDF. The PDF of  $R$  is related to the PDF of  $R_e$  according to

$$f_R(r) = \frac{f_{R_e} \left[ (1/r - 1/r_{\max})^{-1} \right]}{(1 - r/r_{\max})^2}. \quad (1)$$

According to Young-Laplace theory, the capillary pressure head  $h$  and pore radius  $r$  are related according to  $h = \kappa/r$ , where  $\kappa = 2\gamma \cos \alpha / (\rho g)$ ,  $\gamma$  is interface surface tension,  $\alpha$  is the interface contact angle,  $\rho$  is fluid density, and  $g$  is gravitational acceleration. For water in a glass tube,  $\kappa \approx 0.149 \text{ cm}^2$ . It then follows that the PDF of the random capillary pressure head  $H$  is

$$f_H(h) = \frac{f_{R_e} \left[ (h/\kappa - 1/r_{\max})^{-1} \right]}{(h/\kappa - 1/r_{\max})^2}. \quad (2)$$

Further, if  $R_e$  is lognormally distributed, the PDF for the capillary pressure head  $H$  can be written as

$$f_H(h) = \frac{1}{\sqrt{2\pi}\sigma_Z(h - h_c)} \exp \left[ - \left( \frac{\log(h - h_c) - \mu_\eta}{\sqrt{2}\sigma_Z} \right)^2 \right], \quad (3)$$

for all  $h > h_c$ , where  $h_c = \kappa/r_{\max}$  is the bubbling pressure head,  $\mu_\eta = \log(\kappa) - \mu_Z$  is the mean of  $\log(H)$ ,  $\sigma_Z^2$  is the variance of  $Z$ ,  $\mu_Z$  is the mean of  $Z$ , and  $Z = \log(R_e)$ . Kosugi (1994) used the dimensionless random variable  $R'_e = R_e/r_{\max}$  to obtain the PDF in (3) with the parameters  $\mu_Z$  and  $\sigma_Z^2$  scaled appropriately.

As shown by Kosugi (1994), the three-parameter moisture retention curve that follows from (3) is

$$\theta^*(h) = \begin{cases} \frac{1}{2} \operatorname{erfc} \left( \frac{\log(h - h_c) - \mu_\eta}{\sqrt{2}\sigma_Z} \right) & h > h_c, \\ 1 & h \leq h_c, \end{cases} \quad (4)$$

where  $\theta^*(h) = (\theta(h) - \theta_r)/(\theta_s - \theta_r)$  is moisture capacity,  $\theta(h)$  is volumetric moisture content,  $\theta_r$  is residual moisture content,  $\theta_s$  is saturated moisture content, and  $\text{erfc}$  is the complementary error function. Kosugi (1994) did not develop a corresponding closed-form equation for unsaturated hydraulic conductivity.

Mualem (1976) developed a functional relation between unsaturated hydraulic conductivity  $K(\theta^*) = K_s K_r$  and capillary pressure head  $h$

$$K_r(\theta^*) = \sqrt{\theta^*} \left[ \left( \int_0^{\theta^*} \frac{dx}{h(x)} \right) / \left( \int_0^1 \frac{dx}{h(x)} \right) \right]^2, \quad (5)$$

where  $K_r$  and  $K_s$  are relative and saturated hydraulic conductivity and  $x$  is an integration variable. Equation (5) can be rewritten in terms of the PDF of capillary pressure head as

$$K_r(\theta^*) = \sqrt{\theta^*} \left[ \left( \int_h^\infty \frac{f_H(x)}{x} dx \right) / \left( \int_0^\infty \frac{f_H(x)}{x} dx \right) \right]^2. \quad (6)$$

Using the theory of Mualem (1976), Kosugi (1996) developed the two-parameter closed-form equation for unsaturated hydraulic conductivity by forcing  $r_{\max} \rightarrow \infty$ . Kosugi (1996) made this simplification because the theory of Mualem (1976) as given in (6) is not readily amenable to integration when  $r_{\max}$  in the three-parameter lognormal distribution is finite.

We obtain a closed-form expression for unsaturated hydraulic conductivity for finite values of  $r_{\max}$  by modifying (5) due to Mualem (1976) into

$$K_r(\theta^*) = \sqrt{\theta^*} \left[ \left( \int_0^{\theta^*} \frac{dx}{h(x) - h_c} \right) / \left( \int_0^1 \frac{dx}{h(x) - h_c} \right) \right]^2, \quad (7)$$

based on the assumption  $f_H(h)/h \approx f_H(h)/(h - h_c)$ . Using this approximation the closed-form unsaturated hydraulic conductivity expression for the three-parameter lognormal model is

$$K_r(h) \simeq \begin{cases} \sqrt{\theta^*} \left\{ \frac{1}{2} \text{erfc} \left[ \frac{\log(h - h_c) - \mu_Z + \sigma_Z^2}{\sqrt{2}\sigma_Z} \right] \right\}^2 & h > h_c, \\ 1 & h \leq h_c. \end{cases} \quad (8)$$

In the limit as  $r_{\max} \rightarrow \infty$ , (8) reduces to the two-parameter unsaturated hydraulic conductivity expression derived by Kosugi (1996). Figure 1 shows that the truncated lognormal pore-size distribution (8) results in the conductivity curves shifted to the right ( $K_r = 1$  at  $h = h_c$ ) compared to the curves corresponding to the two-parameter model of Kosugi (1996), where  $K_r = 1$  is reached at  $h > 0$ .

## 2.2. Four-parameter lognormal model

To incorporate the lower-tail truncation of Brutsaert (1966) into the distribution of Kosugi (1994), we introduce the random variable  $\hat{R}_e$  defined by

$$\hat{R}_e = \left( \frac{1}{R - r_0} - \frac{1}{r_{\max}} \right)^{-1}, \quad (9)$$

which can be shown to yield the following lognormal PDF for capillary pressure head,

$$f_H(h) = \frac{1}{\sqrt{2\pi}u\sigma_Z} \exp \left[ - \left( \frac{\log(u) - \mu_\eta}{\sqrt{2}\sigma_Z} \right)^2 \right], \quad (10)$$

for all  $h \in [h_c, h_{\max}]$  where  $u = (1/h - 1/h_{\max})^{-1} - h_c$  and  $h_{\max} = \kappa/r_0$  is the pressure head associated with the smallest undrainable pores in the medium corresponding to the physical boundary on pore radius. Figure 2 shows the three lognormal PDFs for capillary pressure head: the classical (non-truncated) lognormal distribution, the upper-truncated lognormal distribution (3), and the doubly truncated lognormal distribution (10). It can be seen in the figure the model of Kosugi (1994) departs from the classical lognormal distribution only at small head values (large pore radii) whereas the proposed four-parameter distribution departs from the classical distribution at both the lower and upper limbs of the function.

The moisture retention curve is derived from (10) in a manner similar to that done by Kosugi (1996) for the four-parameter lognormal distribution, and is given by

$$\theta^*(h) = \begin{cases} \frac{1}{2} \operatorname{erfc} \left[ \frac{\log(u) - \mu_\eta}{\sqrt{2}\sigma_Z} \right] & h_c < h < h_{\max}, \\ 1 & h \leq h_c, \\ 0 & h \geq h_{\max}. \end{cases} \quad (11)$$

Finally, it can be shown that the closed-form expression for unsaturated hydraulic conductivity using the doubly truncated PDF (10) and the modified equation of Mualem (1976) (6) is

$$K_r(h) \simeq \begin{cases} \sqrt{\theta^*} \left\{ \frac{1}{2} \operatorname{erfc} \left[ \frac{\log(u) - \mu_\eta - \sigma_Z^2}{\sqrt{2}\sigma_Z} \right] \right\}^2 & h_c < h < h_{\max}, \\ 1 & h \leq h_c, \\ 0 & h \geq h_{\max}. \end{cases} \quad (12)$$

In the limit as both  $r_{\max} \rightarrow \infty$  and  $r_0 \rightarrow 0$ , (11) and (12) reduce to corresponding two-parameter expressions from Kosugi (1994, 1996).

### 3. Fitting four-parameter lognormal model to experimental data

The four-parameter lognormal model for moisture retention (11) was fitted to experimental data (same data used by van Genuchten (1980) and Kosugi (1996)) using non-linear least squares to estimate the parameters  $r_0$ ,  $r_{\max}$ ,  $\mu_Z$  and  $\sigma_Z^2$ . We also estimated  $\theta_r$  and  $\theta_s$  from experimental data, rather than using reported values. Using estimated parameters in the four-parameter model (12), predictions of unsaturated hydraulic conductivity were compared to measured values. The results for Hygiene sandstone and silt loam G.E. 3 are shown in Figures 3 and 4. A summary of estimated parameters for these soils is given in Table 1. Figure 3(b) shows a solid curve computed using the approximate, but closed-form expression for  $K_r$  (7) and a dashed curve numerically integrated from the traditional Mualem relationship (5), which are nearly identical for this set of parameters. The fits to experimental data are comparable to those of Kosugi (1996) and

van Genuchten (1980). Markov-chain Monte Carlo simulations with the four-parameter model and moisture retention curve data show that the logarithms of parameters  $r_0$  and  $r_{\max}$  are sometimes identifiable only in a threshold sense. No well-defined optimal value exists, the data imply the log minimum pore size is less than a threshold, or the log maximum is greater than a threshold.

The four-parameter lognormal model was also fitted to moisture retention data for Beit Netofa clay, using these estimated parameter values to make predictions of conductivity. The four-parameter model yielded predictions of unsaturated conductivity comparable to the two-parameter models of Kosugi (1996) and van Genuchten (1980) as shown by the dotted curves in Figure 5(a) and (b) (labeled 2- and 3-parameter). However, when hydraulic conductivity and moisture retention data were jointly used to estimate the parameters  $r_0$ ,  $r_{\max}$ ,  $\mu_Z$ ,  $\sigma_Z^2$ ,  $\theta_r$ , and  $\theta_s$ , there was marked improvement in model fits to the data as shown by the solid curves in Figure 5(a) and (b) (labeled 4-parameter). The parameter values used in the fitted model for moisture retention data shown in Figure 5(a) are the same values used for the fitted conductivity model in Figure 5(b). The same procedure did not yield similar results when applied to the two-parameter model of Kosugi (1996), as shown by the dashed curves in plots (a) and (b) of Figure 5, which were obtained by jointly using moisture retention and conductivity data to estimating the parameters  $\mu_Z$  and  $\sigma_Z^2$ , with  $r_{\max} \rightarrow \infty$ .

#### 4. Discussion

The proposed four-parameter lognormal model fits moisture retention data from three representative soils similar to the models of van Genuchten (1980) and Kosugi (1996). Predictions of unsaturated hydraulic conductivity are comparable to those of van Genuchten (1980) and Kosugi (1996) for Hygiene sandstone and silt loam G.E. 3. Estimating model parameters for Beit Netofa clay using only moisture retention data does not yield good predictions of unsaturated hydraulic conductivity (similar to the models of van Genuchten (1980) and Kosugi (1996)). It was essential to use both moisture retention and hydraulic conductivity data to estimate the parameters and improve the fit of the proposed model over the models of Kosugi (1994) and van Genuchten (1980). This indicates moisture retention data alone may not be sufficient to estimate all four (or six, when including  $\theta_r$  and  $\theta_s$ ) model parameters. If conductivity measurements are available, they should be used with moisture retention data, to arrive at a more realistic closed-form model for unsaturated soil moisture retention and hydraulic conductivity.

One would expect a model with a larger number of adjustable parameters to fit observed data better than similar models with less parameters, but an improved fit is often at the expense of the physical significance of the parameters. The proposed four-parameter model does not make its gains in model-data fit at the expense of parameter realism. The parameters follow the philosophy Kosugi (1994) used in deriving his model: they are related to pore-size distribution statistics, rather than being fitting parameters. The development and use of the modified model of Mualem (1976) enables one to obtain approximate but closed-form expressions for

unsaturated hydraulic conductivity for both the three- and four-parameter lognormal pore size distributions. The model derived here is a generalization of the the lognormal model of Kosugi (1996) for non-zero minimum pore radius and non-infinite maximum pore radius.

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Table 1: Estimated parameter values for proposed 4-parameter model

Medium	$r_0(\text{m})$	$r_{\max}(\text{m})$	$\mu_Z$	$\sigma_Z$	$\theta_s$	$\theta_r$
Hygiene Sandstone	$1.07 \times 10^{-4}$	$2.52 \times 10^{-3}$	-6.300	0.337	0.250	0.153
Silt Loam G.E. 3	$1.48 \times 10^{-4}$	$1.27 \times 10^{-2}$	-7.927	1.118	0.395	0.192
Beit Netofa Clay	$4.36 \times 10^{-7}$	$1.32 \times 10^{-2}$	-11.06	2.333	0.450	0.100

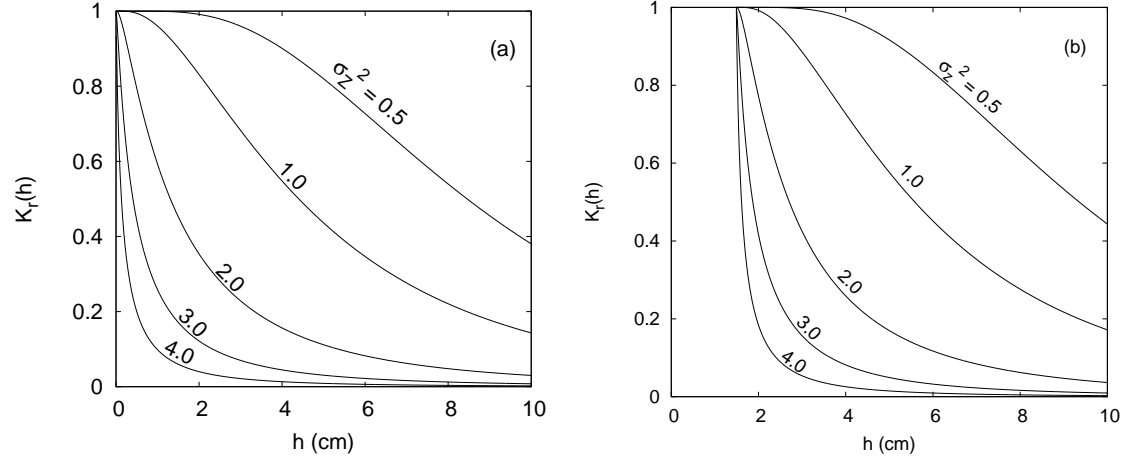


Figure 1: Predicted  $K_r(h)$  for (a) the two-parameter model of Kosugi (1996), and (b) the proposed three-parameter lognormal model for finite  $r_{\max}$ .

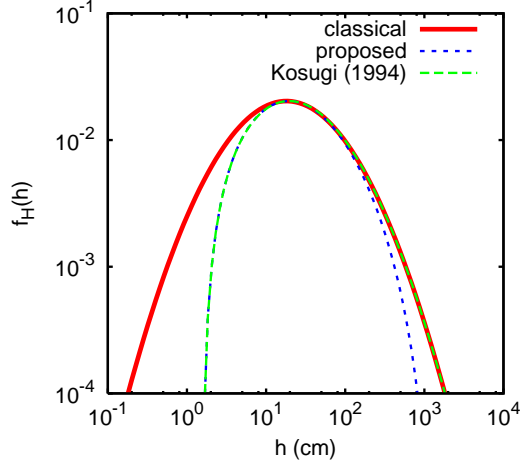


Figure 2: Comparison of lognormal capillary pressure head PDFs for the classical, three-parameter (Kosugi, 1994), and proposed four-parameter (10) models.

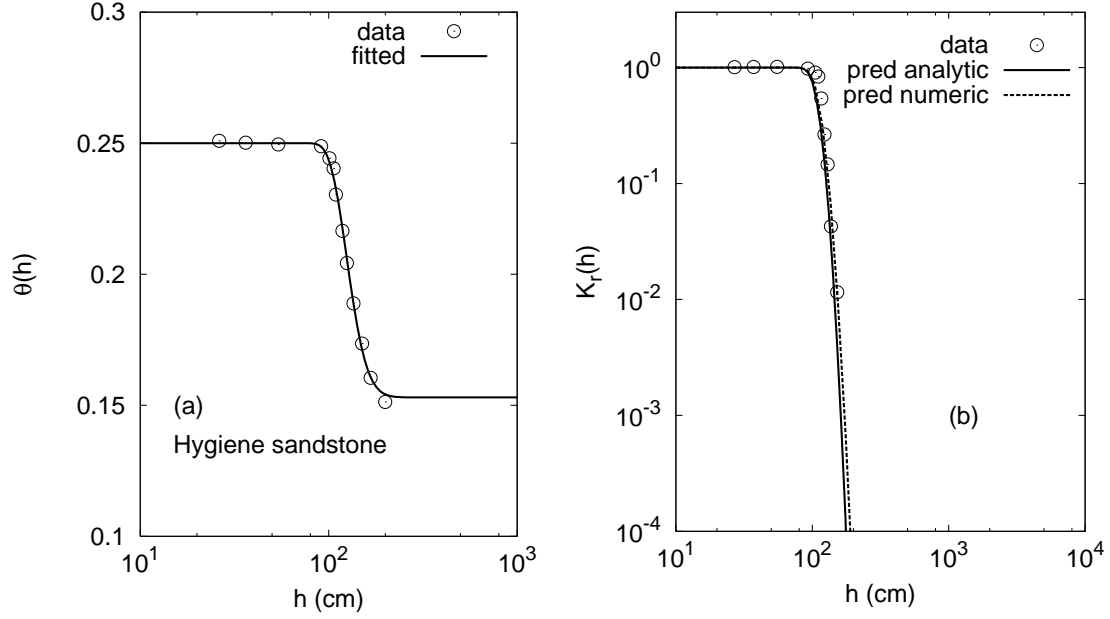


Figure 3: (a) Moisture content data and fitted moisture retention curve, and (b) measured and predicted  $K_r$  (using both analytical and numerical integration) for Hygiene Sandstone (Mualem (1976) soil index 4130).



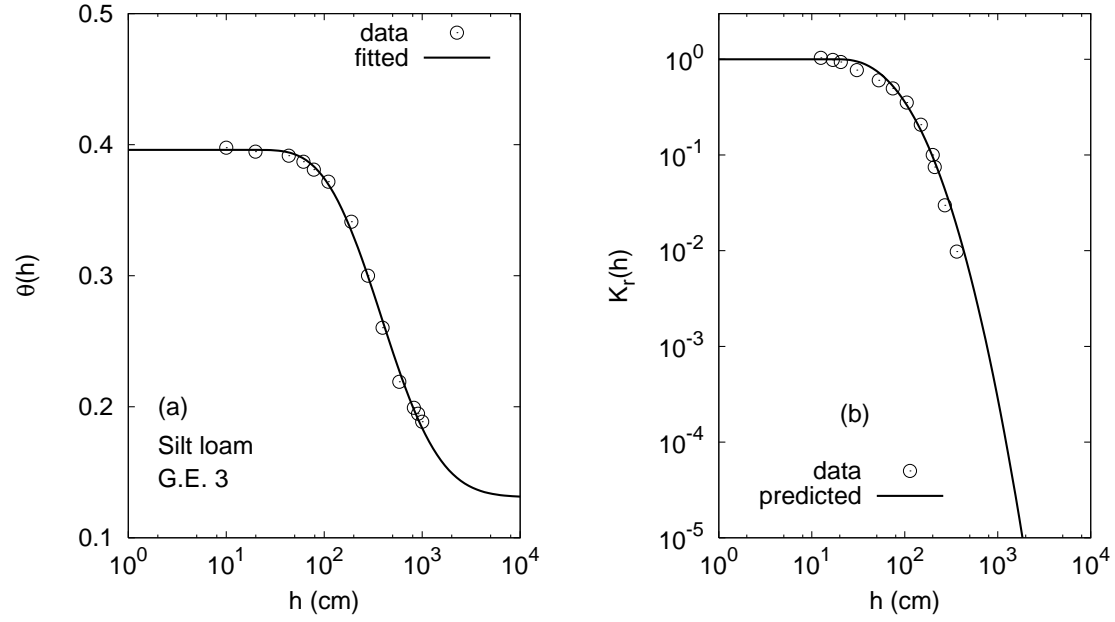


Figure 4: (a) Moisture content data and fitted moisture retention curve, and (b) measured and predicted  $K_r$  for Silt Loam G.E. 3 (Mualem (1976) soil index 3310).

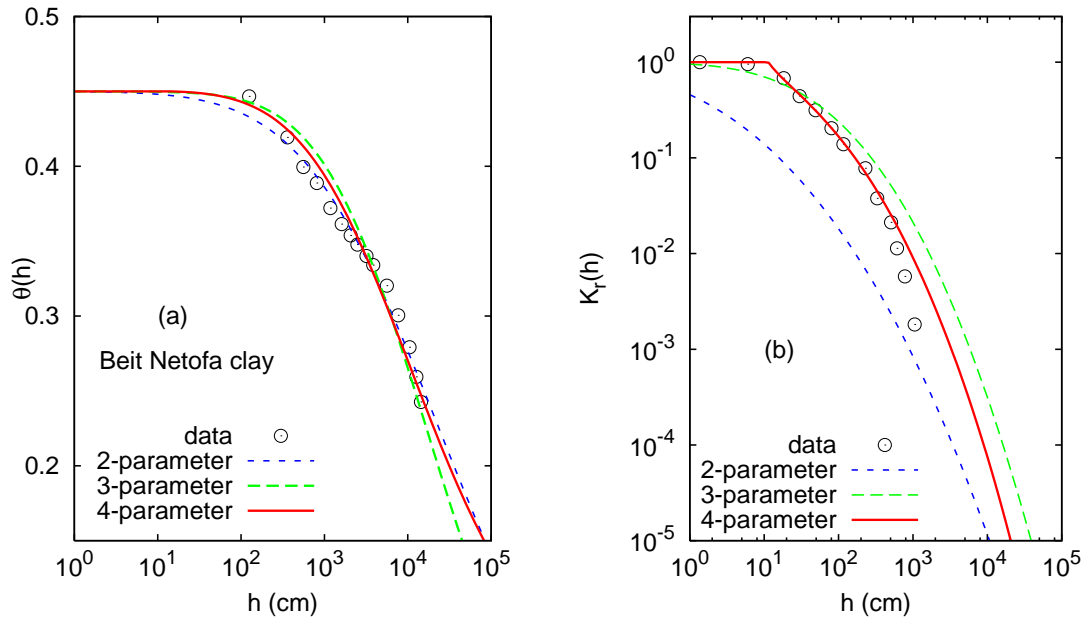


Figure 5: (a) Moisture content data and fitted moisture retention curve, and (b) measured and fitted  $K_r$  for Beit Netofa Clay (Mualem (1976)) soil index 1006).

